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Energy Cascades and Coherent Structures in Geophysical Turbulence: a statistical mechanics approach

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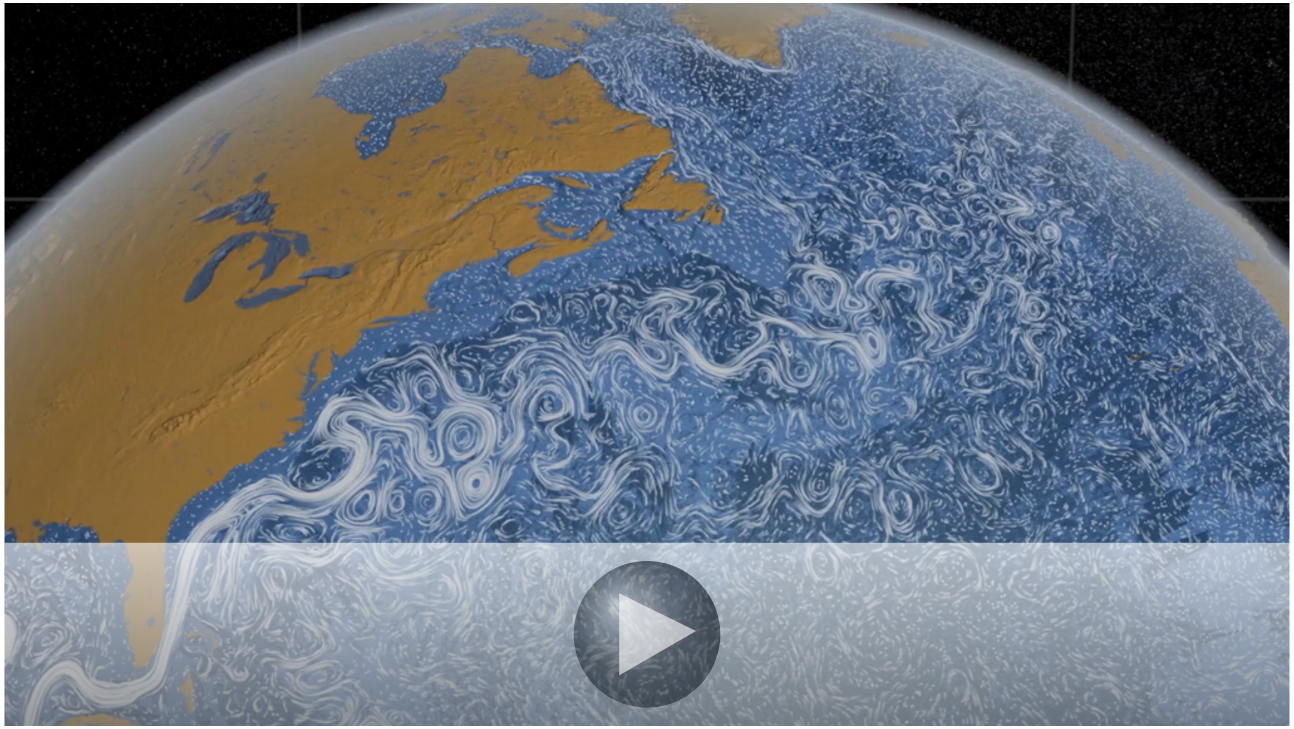
University of Rome, *Tor Vergata*

Motivation: Currents and structures in the ocean

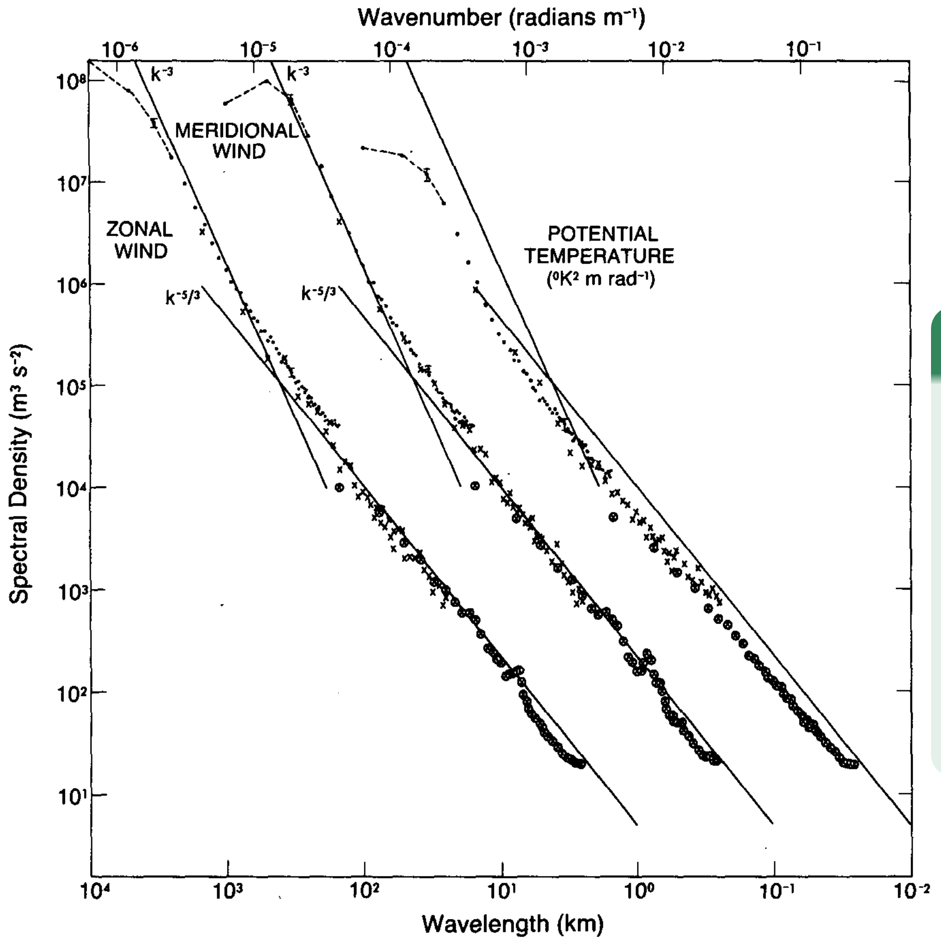
Ocean Surface Circulation reconstruction from satellite observations and in-situ data (NASA Visualization[1])



- *Energy at all scales of motion*. Typical of turbulent flows.
- *Long-lived coherent structures*. Typical of 2D turbulence.

[1] Full-length high-resolution movie: `http://svs.gsfc.nasa.gov/vis/a000000/a003800/a003827/`.

Energy Spectrum in the Atmosphere



Aircraft Measurements[2].

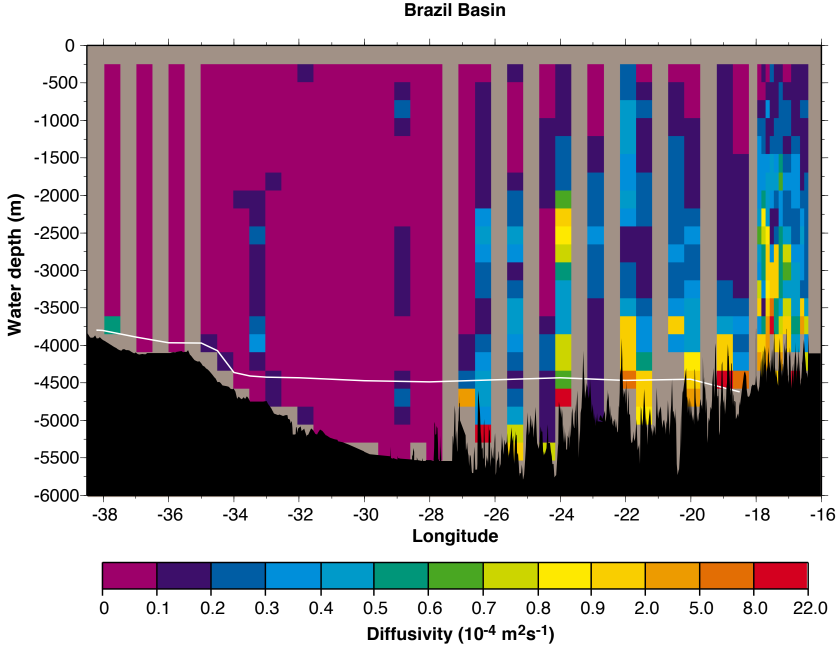
Spectrum Interpretation

- Synoptic Scale k^{-3}: Downscale potential enstrophy cascade.
- Mesoscale $k^{-5/3}$:
  - 2D/QG Upscale energy cascade?
  - 3D Downscale energy cascade?
  - Something else? Gravity waves?

[2] G. D. Nastrom et al. (1984). *Nature*

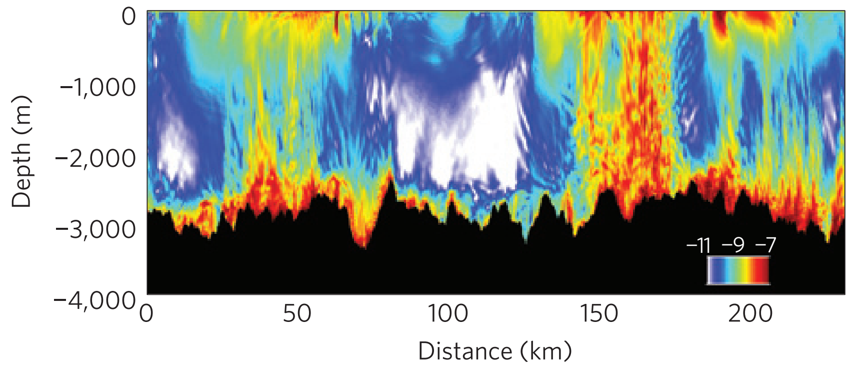
Vertical Mixing and Dissipation in the ocean

Enhanced vertical mixing and dissipation over rough topography:



K. L. Polzin et al. (1997). *Science*

Energy dissipation rates (log):



M. Nikurashin et al. (2013). *Nature Geoscience*

How can we understand the coexistence of 2D (quasi-geostrophic) turbulence at large scales with small-scale mixing and dissipation in terms of energy transfer across scales?

Dynamical Models: 2D/3D Euler flows

Navier-Stokes equations for incompressible turbulent flows:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} P + \nu \Delta \mathbf{u} + \mathbf{F},$$
$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0.$$

When $\nu = F = 0$ (no forcing and no dissipation), we have the *Euler equations*.
In terms of *vorticity* $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$,

- **For a 2D domain**, $\boldsymbol{\omega} = \omega \mathbf{n}$, vorticity is conserved along Lagrangian trajectories:
$$\partial_t \omega + \mathbf{u} \cdot \boldsymbol{\nabla} \omega = 0.$$
- **For a 3D domain**, it is not:
$$\partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \boldsymbol{\nabla} \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla} \mathbf{u}.$$

Dynamical Models: Geophysical flows

Geophysical flows are subjected to additional forces: Coriolis force (rotation) and buoyancy (density stratification).

- **3D Boussinesq flows:**

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P - 2\Omega\mathbf{e_z}\times\mathbf{u} - N\theta\mathbf{e_z},$$
$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta = Nu_z,$$
$$\nabla\cdot\mathbf{u} = 0.$$

Potential vorticity $\Pi = f\partial_z\theta - N\omega_z + \boldsymbol{\omega}\cdot\nabla\theta$ is conserved: $\partial_t\Pi + \mathbf{u}\cdot\nabla\Pi = 0$.

- In the asymptotic regime of *strong rotation and stratification*, **quasi-geostrophic equations**[3]:

$$\partial_t q + \mathbf{u}\cdot\nabla q = 0,$$
$$q = -\Delta\psi + f + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right).$$

This regime describes well the large scales of the atmosphere and oceans.

[3] G. K. Vallis (2006). *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-scale Circulation*. Cambridge University Press

Hamiltonian Structure

Inviscid ($\nu = 0$) 2D/3D and geophysical flows have a Hamiltonian structure[4]: there exist a Hamiltonian functional $\mathcal{H}$ and a Poisson bracket $\{\cdot, \cdot\}$ such that the dynamics has a form analogous to $\dot{x}_i = \{x_i, \mathcal{H}\}$.

E.g. 2D Turbulence: Stream function ψ such that $\omega = -\Delta\psi$, $\mathcal{H} = \int \omega\psi/2$.

$$\dot{\omega} = -\mathbf{u} \cdot \nabla\omega = \partial(\omega, \psi) = \mathscr{D}(\omega)\frac{\delta\mathcal{H}}{\delta\omega}, \qquad \text{with } \mathscr{D}(\omega) = \partial(\omega, \cdot).$$

[4] R. Salmon (1988). *Ann. Rev. Fluid Mech.* P. J. Morrison (1998). *Rev. Mod. Phys.*

Hamiltonian Structure

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Consequences of the Poisson structure degeneracies

The operator $\mathscr{D}$ is degenerate:

- **Infinity of steady states.** For 2D/QG flows, $\omega = F(\psi)$, for an arbitrary function F.
- **Additional conserved quantities:**
  - 3D flows: *Helicity*[4] $\int \boldsymbol{\omega}\cdot\mathbf{u}$.
  - 2D/QG/Boussinesq flows: *Casimir invariants* $\int s(\omega)$ (or replace vorticity ω by potential vorticity q or Π). In particular, there is a second quadratic invariant, the (potential) *enstrophy* $\int \omega^2$.
    Equivalently, the area $\gamma(\sigma)$ occupied by a vorticity level σ is conserved.

[4] H. K. Moffatt (1969). *J. Fluid Mech.* D. Serre (1984). *Physica D*

- Hamiltonian systems
- Huge number of degrees of freedom: $\sim Re^{9/4}$

Calls for a statistical mechanics approach

But

- Infinite dimensional phase space
- Infinite number of conservation laws
- (Long-range interactions)

Outline

Canonical distribution for Galerkin-truncated 2D flows

Truncated 2D Euler: $\mathcal{B} = \{\mathbf{k} \in \mathbb{Z}^2, k_{\mathrm{min}} \leq k \leq k_{\mathrm{max}}\}$.

Energy and Enstrophy:

$$\mathcal{E}[\omega] = \frac{1}{2}\int_{\mathcal{D}} \omega\psi = \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{B}} \frac{|\omega_{\mathbf{k}}|^2}{k^2},$$

$$\mathcal{G}_2[\omega] = \frac{1}{2}\int_{\mathcal{D}} \omega^2 = \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{B}} |\omega_{\mathbf{k}}|^2.$$

Detailed Liouville theorem:

$$\frac{\partial \dot{\omega}_{\mathbf{k}}}{\partial \omega_{\mathbf{k}}} = 0, \text{ and therefore } \sum_{\mathbf{k}\in\mathcal{B}} \frac{\partial \dot{\omega}_{\mathbf{k}}}{\partial \omega_{\mathbf{k}}} = 0.$$

Any measure of the form $\mu(d\omega) = \rho(\mathcal{E}[\omega], \mathcal{G}_2[\omega])d\omega$ is an invariant measure.

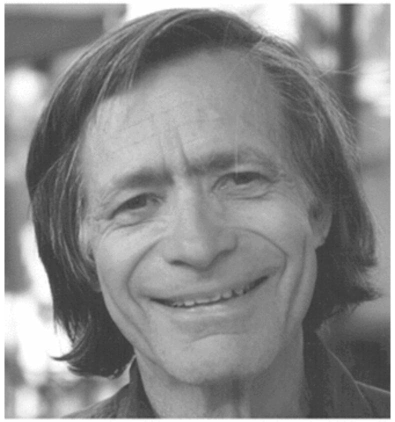
Canonical probability density[5]:

$$\rho(\{\omega_{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{B}}) = \frac{1}{\mathcal{Z}} e^{-\beta\mathcal{E}[\omega]-\alpha\mathcal{G}_2[\omega]},$$

$$= \frac{1}{\mathcal{Z}} e^{-\sum_{\mathbf{k}\in\mathcal{B}}(\beta+\alpha k^2)\frac{|\omega_{\mathbf{k}}|^2}{2k^2}},$$

with $\mathcal{Z}$ the *partition function*:

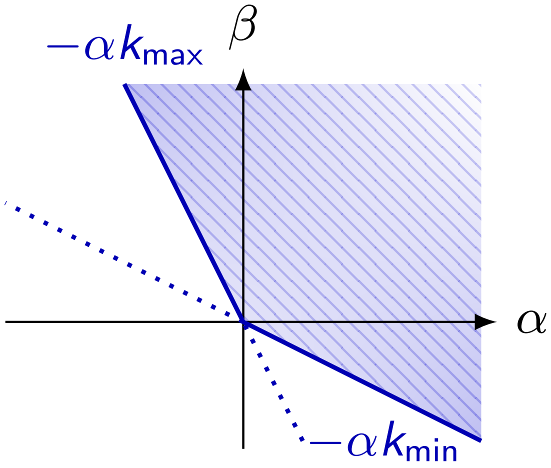
$$\mathcal{Z} = \int e^{-\sum_{\mathbf{k}\in\mathcal{B}}(\beta+\alpha k^2)\frac{|\omega_{\mathbf{k}}|^2}{2k^2}} \prod_{\mathbf{k}\in\mathcal{B}} d\omega_{\mathbf{k}} = \prod_{\mathbf{k}\in\mathcal{B}} \sqrt{\frac{2\pi k^2}{\beta+\alpha k^2}}.$$



Robert H. Kraichnan (1928–2008)

[5] R. H. Kraichnan (1967). *Phys. Fluids*; R. H. Kraichnan (1975). *J. Fluid Mech.* R. H. Kraichnan and D. C. Montgomery (1980). *Rep. Prog. Phys.*

Canonical distribution for Galerkin-truncated 2D flows

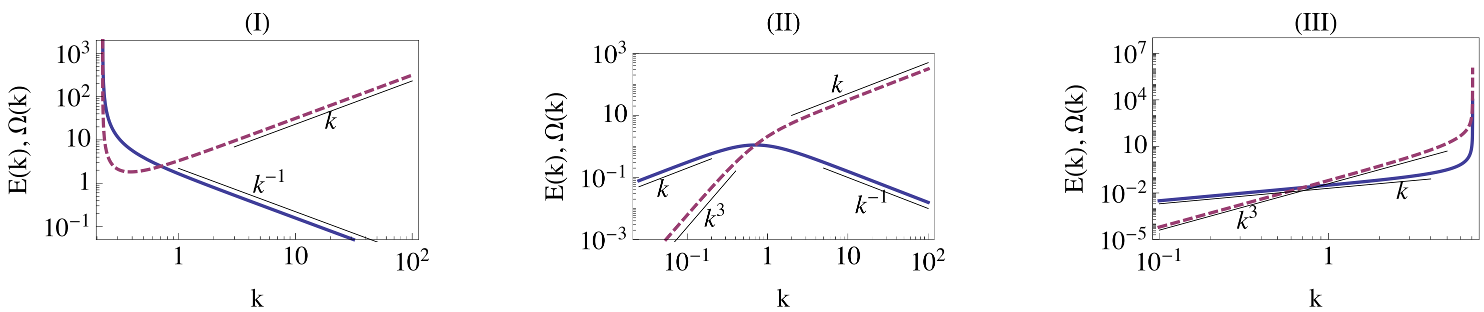


Thermodynamic space Realizability condition:
$\forall \mathbf{k} \in \mathcal{B}, \beta + \alpha k^2 > 0$.

1. $\alpha > 0, \beta < 0$: High-energy regime.
2. $\alpha > 0, \beta > 0$: Intermediate regime.
3. $\alpha < 0, \beta > 0$: High-enstrophy regime.

Equilibrium Energy spectra

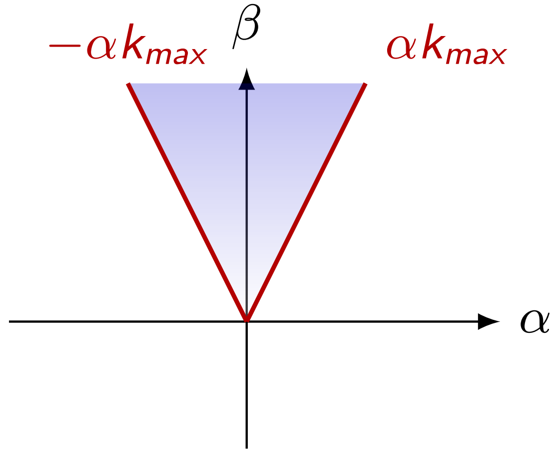
$$\langle E \rangle = -\frac{\partial \ln \mathcal{Z}}{\partial \beta} = \frac{1}{2} \sum_{\mathbf{k} \in \mathcal{B}} \frac{1}{\beta + \alpha k^2}; \langle E(k) \rangle = \frac{\pi k}{\beta + \alpha k^2}, \langle E \rangle = \int_0^{+\infty} \langle E(k) \rangle dk.$$



Infrared divergence in the $\beta < 0$ regime. Inverse cascade for 2D Turbulence.

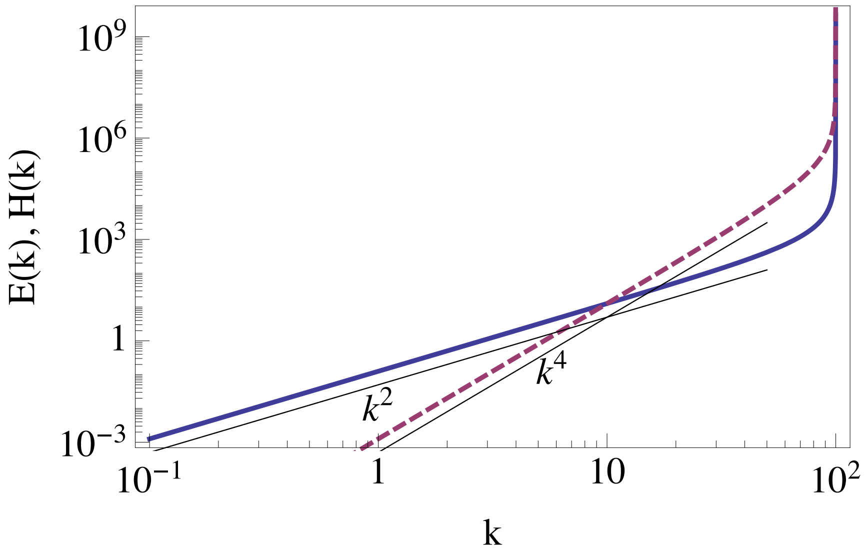
What about 3D Turbulence?

The Liouville theorem stills holds[5]. Canonical probability density:



$$\rho(\{u_+(\mathbf{k}), u_-(\mathbf{k})\}) = \frac{1}{\mathcal{Z}} e^{-\beta E - \alpha H},$$
$$= \frac{1}{\mathcal{Z}} e^{-\sum_{\mathbf{k}}[(\beta+\alpha k)|u_+(\mathbf{k})|^2 + (\beta-\alpha k)|u_-(\mathbf{k})|^2]}.$$

Partition Function: $\mathcal{Z} = \prod_{\mathbf{k}} \frac{2\pi}{\sqrt{\beta^2-\alpha^2 k^2}}$. $\beta > |\alpha| k_{\mathrm{max}} > 0$.

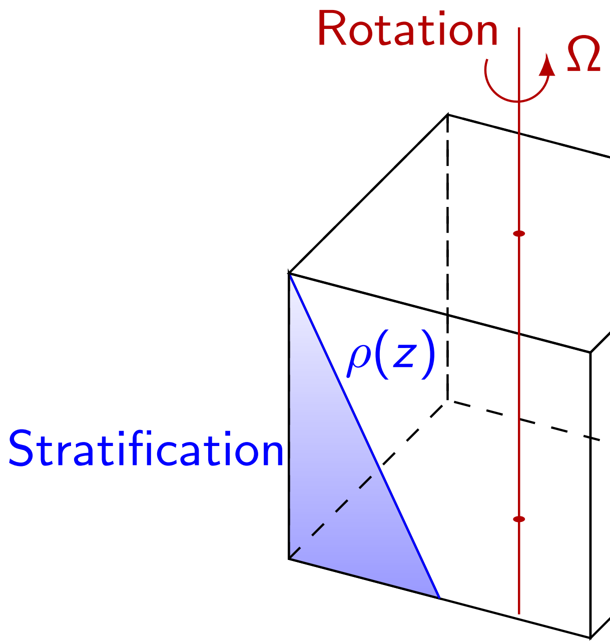


$$\langle E \rangle = -\frac{\partial \ln \mathcal{Z}}{\partial \beta},$$
$$= \sum_{\mathbf{k}} \frac{\beta}{\beta^2 - \alpha^2 k^2},$$
$$\langle E(k) \rangle = \frac{4\pi\beta k^2}{\beta^2 - \alpha^2 k^2}.$$

Ultraviolet Catastrophe[6]

[5] T. D. Lee (1952). *Q. Appl. Math.*
[6] R. H. Kraichnan (1973). *J. Fluid Mech.*

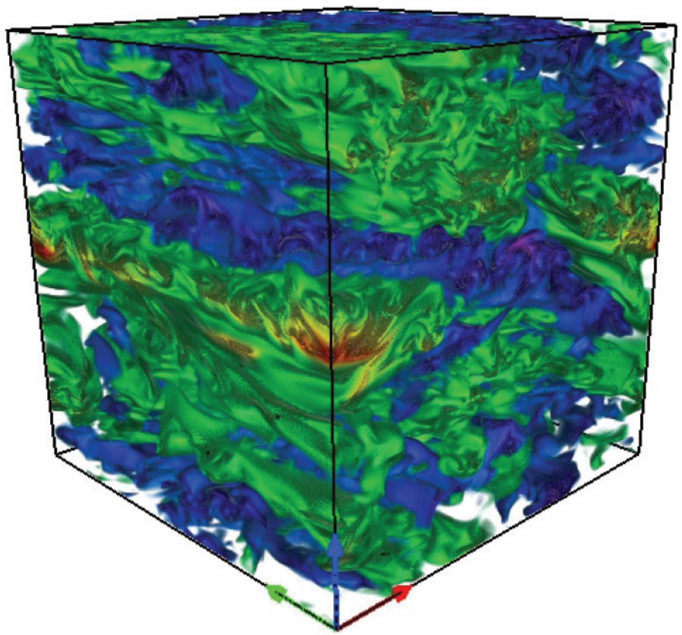
Rotating-Stratified Turbulence: Idealized setup



Boussinesq Equations; Cubic domain with periodic BC.

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P - 2\Omega \mathbf{e_z} \times \mathbf{u} - N\theta \mathbf{e_z},$$
$$\partial\theta + \mathbf{u}\cdot\nabla\theta = N u_z,$$
$$\nabla\cdot\mathbf{u} = 0.$$

DNS[7] (512^3, Re $\sim 10^4$), θ:



Fr $= 0.1$, Ro $= 0.4$

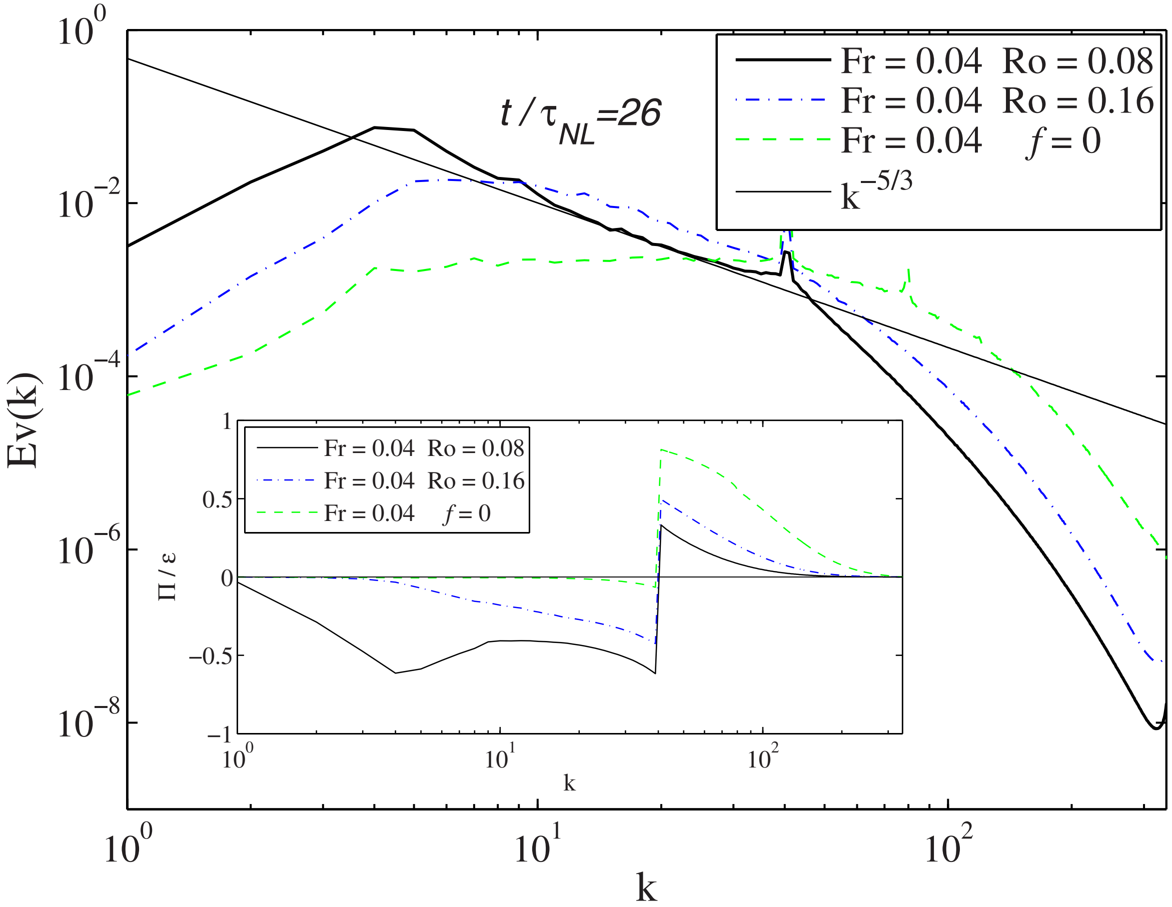
Non-dimensional numbers

- Stratification: $\mathrm{Fr} = \frac{U}{NL}$
- Rotation: $\mathrm{Ro} = \frac{U}{fL}$, $(f = 2\Omega)$

[7] R. Marino et al. (2013a). *Phys. Rev. E*

Energy spectrum and fluxes

Kinetic energy spectrum and fluxes, DNS (1024^3, Re $\approx 10^3$, $k_f = 40$) of stratified flows with or without rotation[8]:

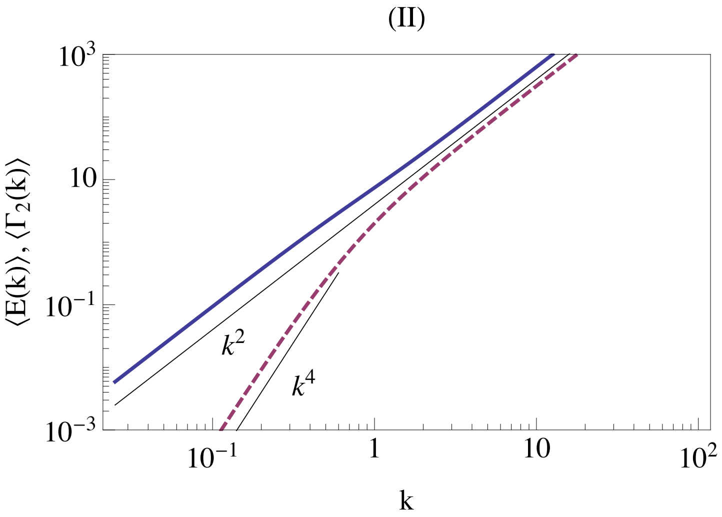


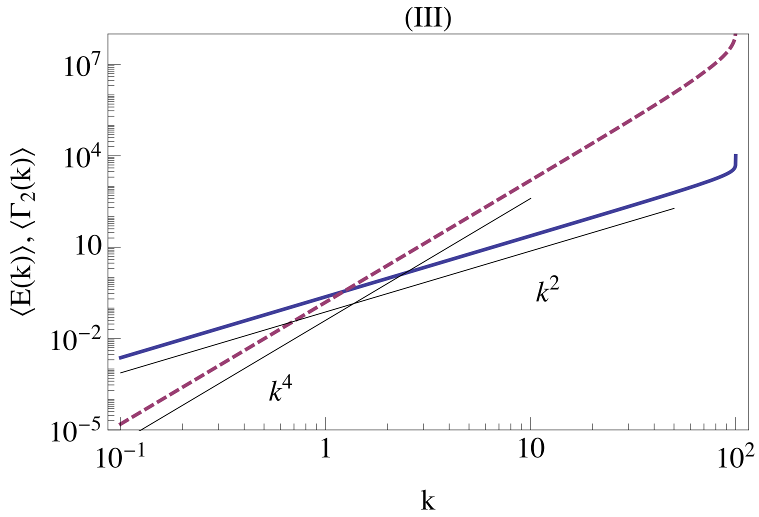
Seamless transition from upscale to downscale energy cascade as rotation weakens (Ro increases).

[8] R. Marino et al. (2013b). *Europhys. Lett.*

Absolute Equilibrium

Rotating-Stratified flows at absolute equilibrium[9]:





In the two possible regimes ($\beta > 0$), the energy at equilibrium is close to equipartition, with a possible divergence at small scales (ultraviolet catastrophe), like in 3D turbulence, which points at a *downscale cascade of energy*.

[9] P. Bartello (1995). *J. Atmos. Sci.* see M. L. Waite and P. Bartello (2004). *J. Fluid Mech.* for the purely stratified case.

Normal modes of the linearized equations

Linearized Boussinesq dynamics in Fourier space[10]:

$$\dot{\mathbf{Z}}(\mathbf{k}) = \mathbf{L}(\mathbf{k})\mathbf{Z}(\mathbf{k}), \quad \text{with } \mathbf{Z}(\mathbf{k}) = \begin{pmatrix} \hat{\omega}_{\parallel}(\mathbf{k}) \\ -ik\hat{u}_{\parallel}(\mathbf{k}) \\ -k_{\perp}\hat{\theta}(\mathbf{k}) \end{pmatrix},$$

$$\mathbf{L}(\mathbf{k}) = \begin{pmatrix} 0 & -f\frac{k_{\parallel}}{k} & 0 \\ f\frac{k_{\parallel}}{k} & 0 & -iN\frac{k_{\perp}}{k} \\ 0 & -iN\frac{k_{\perp}}{k} & 0 \end{pmatrix}, \quad {}^t\overline{\mathbf{L}(\mathbf{k})} = -\mathbf{L}(\mathbf{k}).$$

$$\mathrm{Sp}\,\mathbf{L}(\mathbf{k}) = \{0, i\sigma(\mathbf{k}), -i\sigma(\mathbf{k})\}, \text{ with } \sigma(\mathbf{k}) = k^{-1}\sqrt{f^2k_{\parallel}^2 + N^2k_{\perp}^2}.$$

Eigenmodes

- Two *inertia-gravity wave modes* $\mathbf{Z}_{\pm}(\mathbf{k})$, $\mathbf{L}(\mathbf{k})\mathbf{Z}_{\pm}(\mathbf{k}) = \pm i\sigma(\mathbf{k})$.
- One *slow mode* $\mathbf{Z}_0(\mathbf{k})$ with zero linear frequency: $\mathbf{L}(\mathbf{k})\mathbf{Z}_0(\mathbf{k}) = 0$.

$\mathbf{Z}(\mathbf{k}) = a_0(\mathbf{k})\mathbf{Z}_0(\mathbf{k}) + a_-(\mathbf{k})\mathbf{Z}_-(\mathbf{k}) + a_+(\mathbf{k})\mathbf{Z}_+(\mathbf{k})$

Slow manifold: $a_+(\mathbf{k}) = a_-(\mathbf{k}) = 0$.

[10] C. E. Leith (1980). *J. Atmos. Sci.* P. Bartello (1995). *J. Atmos. Sci.*

Properties of slow modes

Slow manifold and balanced motion

- For rotating-stratified flows: The slow modes are in *hydrostatic balance*: $\partial_z P = -\rho g$, and *geostrophic balance*: $\nabla_\perp P = -2\mathbf{\Omega} \times \mathbf{u}$.
- For rotating flows, the slow modes, are in *geostrophic balance*.
- For stratified flows, the slow modes are *not in hydrostatic balance*, unless $k_\perp = 0$ (*vertically sheared* modes).

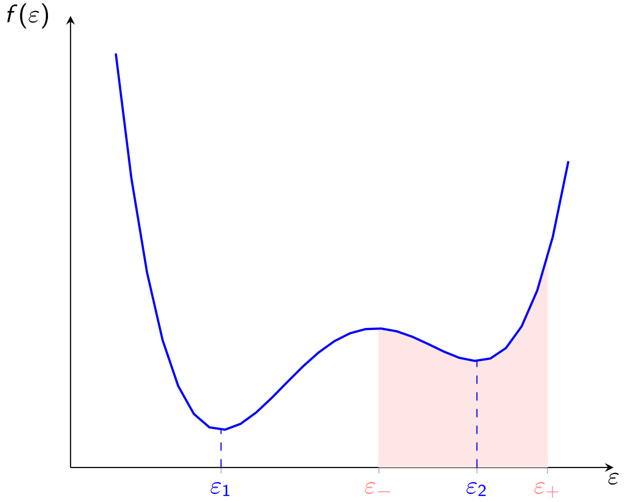
Slow manifold and potential enstrophy

Potential vorticity $\Pi = f\partial_\parallel \theta - N\omega_\parallel + \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\theta$, potential enstrophy $\int \Pi^2$ is a global invariant. Quadratic part Γ_2:

$$\Gamma_2 = \frac{1}{2}\int (f\partial_\parallel \theta - N\omega_\parallel)^2 = \frac{1}{2}\sum_{\mathbf{k}\in\mathcal{B}} \frac{k^2\sigma(\mathbf{k})^2}{k_\perp^2} |a_0(\mathbf{k})|^2$$

For stratified flows, the only modes which carry PV have $k_\perp \neq 0$.

Restricted partition function (General Idea)



Metastable states (local minima of the free energy $f(\varepsilon)$): restrict the integral defining the partition function to a subset Λ' of phase space[11].

Absolute equilibrium:

$$\mathcal{Z}(\beta) = \int_\Lambda e^{-\beta N h(x)} \mu(dx), = \int_0^{+\infty} e^{-\beta N \varepsilon} \Omega(\varepsilon) d\varepsilon, \sim e^{-N\phi(\beta)},$$

Restricted equilibrium:

$$\mathcal{Z}'(\beta) = \int_{\Lambda'} e^{-\beta N h(x)} \mu(dx), = \int_{\varepsilon_-}^{\varepsilon_+} e^{-\beta N \varepsilon} \Omega(\varepsilon) d\varepsilon, \sim e^{-N\phi'(\beta)},$$

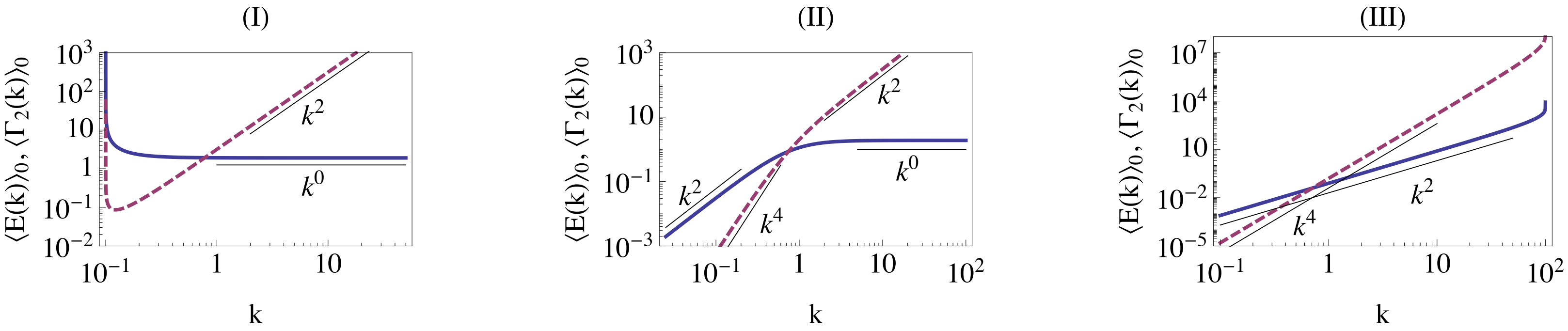
$$\phi(\beta) = \min_{\varepsilon \in \mathbb{R}_+} (\beta\varepsilon - s(\varepsilon)) = \beta\varepsilon_1 - s(\varepsilon_1).$$

$$\phi'(\beta) = \min_{\varepsilon \in [\varepsilon_-, \varepsilon_+]} (\beta\varepsilon - s(\varepsilon)) = \beta\varepsilon_2 - s(\varepsilon_2).$$

[11] O. Penrose and J. L. Lebowitz (1971). *J. Stat. Phys.* O. Penrose and J. L. Lebowitz (1979). In: *Fluctuation Phenomena*. Ed. by E. W. Montroll and J. L. Lebowitz. Amsterdam: North-Holland

Restricted partition function (Results)[12]

- Rotating-Stratified flows at restricted equilibrium (slow manifold only):



$\beta < 0$ regime (I): infrared divergence of the restricted equilibrium energy spectrum, like in 2D turbulence, which points at the existence of an inverse cascade.

[12] C. Herbert et al. (submitted). *J. Fluid Mech.* arXiv:1401.2103.

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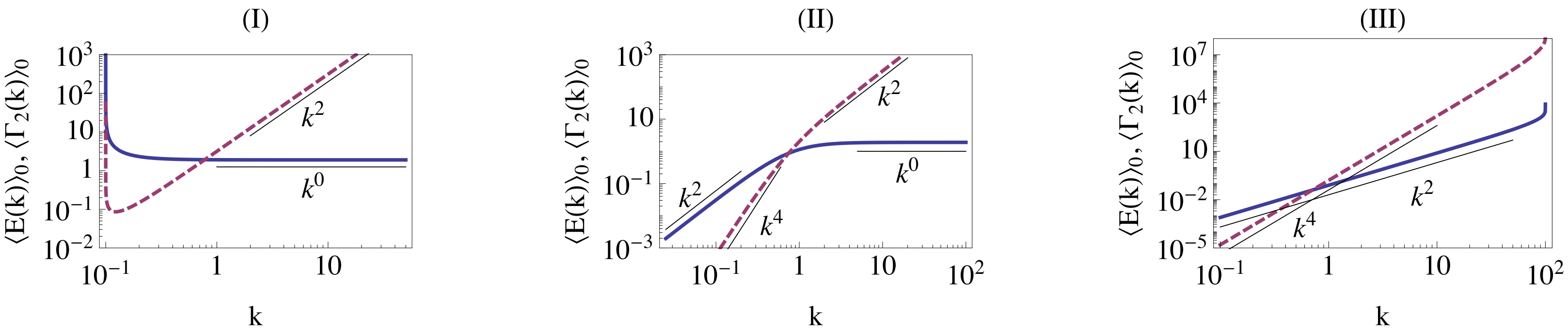

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- Purely rotating flows at restricted equilibrium:
  $\beta < 0$ states still exist: inverse cascade of energy by the vortical modes (i.e. 2D modes).

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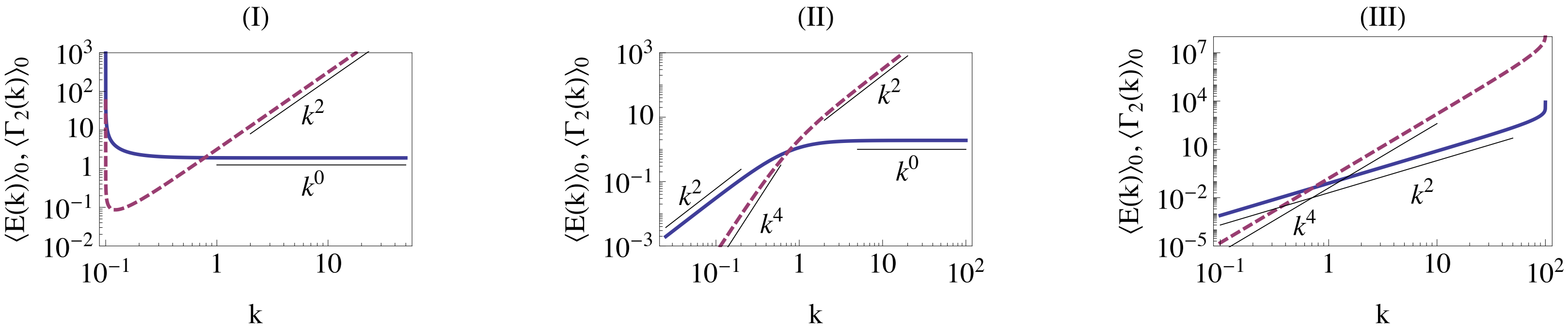


$\beta < 0$ regime (I): infrared divergence of the restricted equilibrium energy spectrum, like in 2D turbulence, which points at the existence of an inverse cascade.

- Purely rotating flows at restricted equilibrium:
  $\beta < 0$ states still exist: inverse cascade of energy by the vortical modes (i.e. 2D modes).
- Purely stratified flows at restricted equilibrium:
  $\beta > 0$ (regimes (II) and (III)): forward energy cascade.

[12] C. Herbert et al. (submitted). *J. Fluid Mech.* arXiv:1401.2103.

Outline

Dynamical Model

Large scales of geophysical flows well described by *quasi-geostrophic equations*.
For pedagogical reasons, we use the Euler equations.
2D Euler equation on a domain $\mathcal{D}$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P,$$
$$\nabla \cdot \mathbf{u} = 0.$$

In terms of vorticity $\partial_t \omega + \mathbf{u} \cdot \nabla \omega = 0$.
Energy:

$$\mathcal{E}[\omega] = \frac{1}{2} \int_{\mathcal{D}} \omega(\mathbf{r}) \psi(\mathbf{r}) d\mathbf{r}, \text{ with } \omega = -\Delta \psi.$$

Casimir Invariants:

$$\mathcal{C}_s[\omega] = \int_{\mathcal{D}} s(\omega(\mathbf{r})) d\mathbf{r}, \qquad \mathcal{G}_n[\omega] = \int_{\mathcal{D}} (\omega(\mathbf{r}))^n d\mathbf{r},$$
$$A(\sigma) = \int_{\mathcal{D}} \Theta(\omega(\mathbf{r}) - \sigma) d\mathbf{r}, \qquad \gamma(\sigma) = \frac{1}{|\mathcal{D}|} \frac{dA}{d\sigma}.$$

Topological constraints?
Multiple Stable Steady States: $\omega = F(\psi)$. How do we select F?

Constructing the microcanonical measure

To build the microcanonical measure, there are two difficulties:

(i) phase space is infinite-dimensional

(ii) there is an infinite number of constraints.

Formally, we define the *microcanonical measure* as

$$\mu_{E,(\Gamma_n)_{n\in\mathbb{N}}}(d\omega) = \frac{1}{\Omega(E,(\Gamma_n)_{n\in\mathbb{N}})}\delta(\mathcal{E}[\omega]-E)\prod_{k=1}^{+\infty}\delta(\mathcal{G}_k[\omega]-\Gamma_k)\prod_{i=1}^{+\infty}d\omega_i.$$

In fact, it should be defined as a limit measure; e.g. by introducing a finite lattice[13] or a finite number of Laplacian eigenmodes[14], and conserving a finite number of constraints.

- Monte-Carlo methods
- Mean-field theory:
  - Simpler expression for $\mu_{E,(\Gamma_n)_{n\in\mathbb{N}}}$.
  - Discussing individual macrostates rather than the ensemble[15].

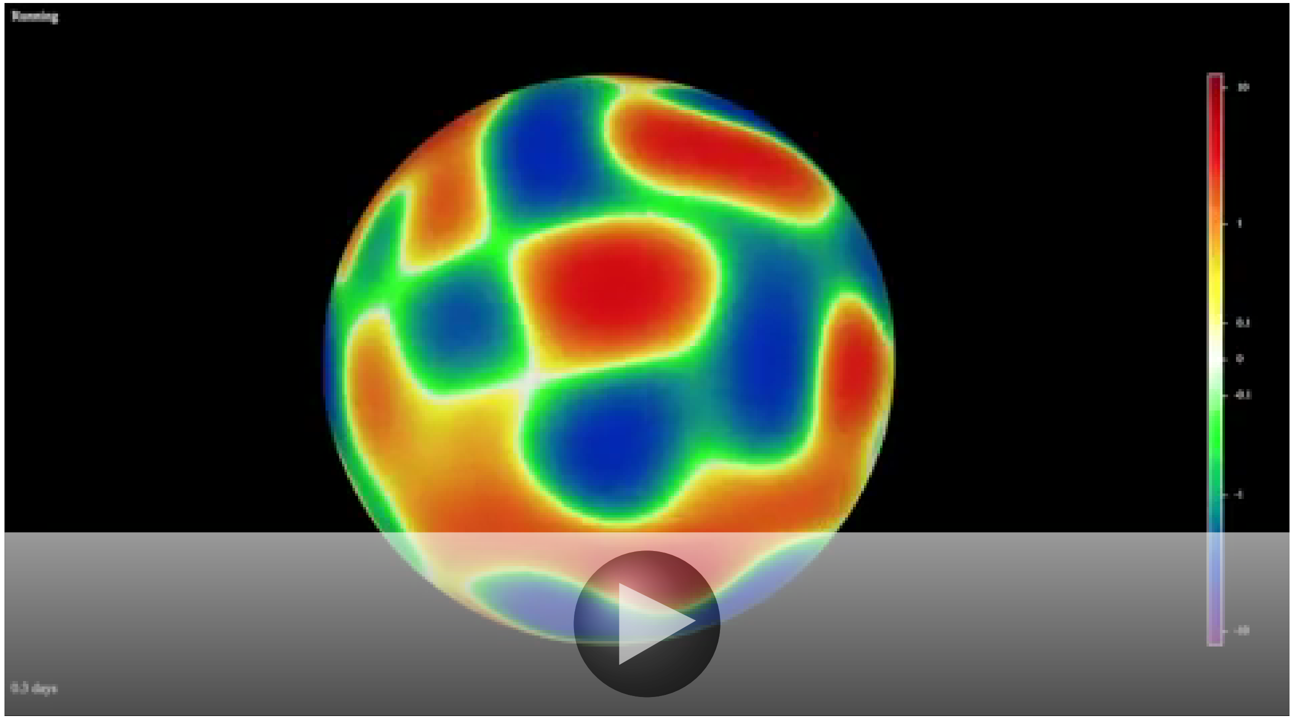
[13] see e.g. M. Potters et al. (2013). *J. Stat. Mech.*

[14] see e.g. F. Bouchet and M. Corvellec (2010). *J. Stat. Mech.*

[15] M. K. H. Kiessling (2008). *AIP Conf. Proc.*

The Phenomenology of the mean-field theory

Small-scale vorticity is mixed by the flow while large-scale coherent structures form.



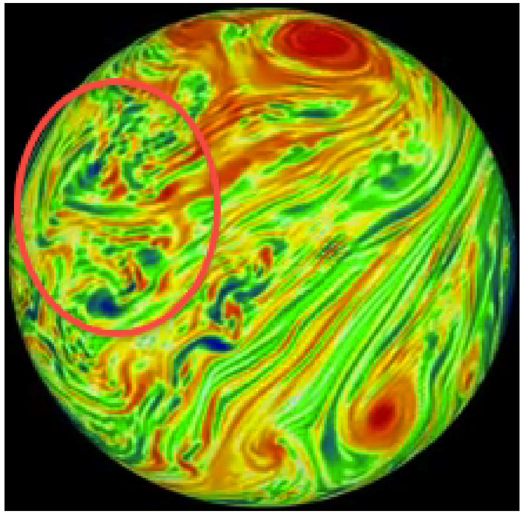
Direct Numerical Simulation[16]: Vorticity Contours.

[16] B. Marston (2011). *Physics*; W. Qi and J. B. Marston (to appear). *J. Stat. Mech.*

The Phenomenology of the mean-field theory

Two levels of description[17]:

- Microstates: fine-grained vorticity field $\omega(\mathbf{x})$.

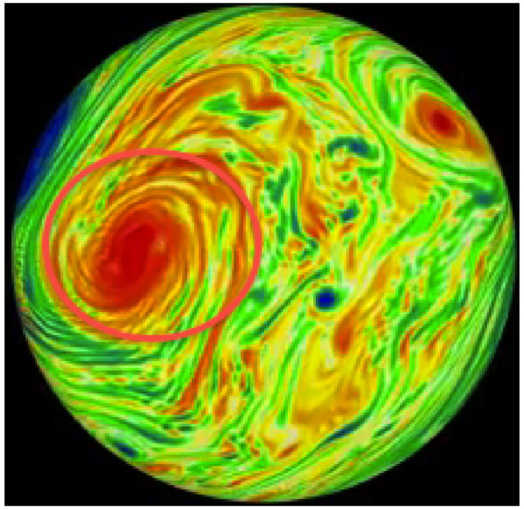


[17] R. Robert and J. Sommeria (1991). *J. Fluid Mech.* R. Robert (1991). *J. Stat. Phys.* J. Miller (1990). *Phys. Rev. Lett.* J. Miller et al. (1992). *Phys. Rev. A*

The Phenomenology of the mean-field theory

Two levels of description[17]:

- Microstates: fine-grained vorticity field $\omega(\mathbf{x})$.
- Macrostates: fine-grained vorticity probability distribution $\rho(\sigma, \mathbf{x})$, $\int \rho(\sigma, \mathbf{x}) d\sigma = 1$.
  Mean coarse-grained vorticity: $\overline{\omega}(\mathbf{x}) = \int \sigma \rho(\sigma, \mathbf{x}) d\sigma$.

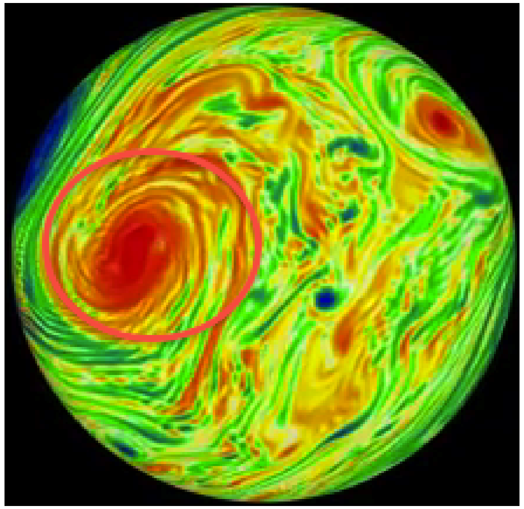


[17] R. Robert and J. Sommeria (1991). *J. Fluid Mech.* R. Robert (1991). *J. Stat. Phys.* J. Miller (1990). *Phys. Rev. Lett.* J. Miller et al. (1992). *Phys. Rev. A*

The Phenomenology of the mean-field theory

Two levels of description[17]:

- Microstates: fine-grained vorticity field $\omega(\mathbf{x})$.
- Macrostates: fine-grained vorticity probability distribution $\rho(\sigma, \mathbf{x})$, $\int \rho(\sigma, \mathbf{x}) d\sigma = 1$.
  Mean coarse-grained vorticity: $\overline{\omega}(\mathbf{x}) = \int \sigma \rho(\sigma, \mathbf{x}) d\sigma$.

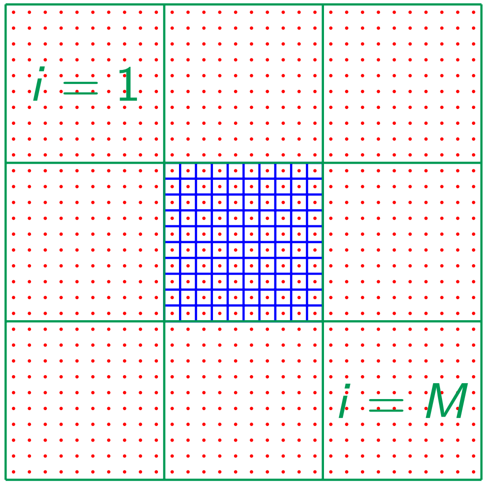


We are going to see how to obtain the most probable macrostate $\rho(\sigma, \mathbf{x})$. This allows us to define the set of *equilibrium states*, a subclass of the steady-states of the Euler equations, through averaging: $\overline{\omega} = F_{E,\gamma(\sigma)}(\overline{\psi})$.

[17] R. Robert and J. Sommeria (1991). *J. Fluid Mech.* R. Robert (1991). *J. Stat. Phys.* J. Miller (1990). *Phys. Rev. Lett.* J. Miller et al. (1992). *Phys. Rev. A*

The mean-field approach: counting the microstates

Let us consider a square lattice with N sites, and a "coarse-grained" lattice of M boxes containing $n = N/M$ sites each.



Finite number of vorticity levels $\mathfrak{S} = \{\sigma_1, \dots, \sigma_K\}$: $\gamma(\sigma) = \sum_{k=1}^{K} \gamma_k \delta(\sigma - \sigma_k)$.

- Microstates:

$$\hat{\omega} = (\omega_{i\alpha})_{\substack{1 \le i \le M \\ 1 \le \alpha \le n}} \in \mathfrak{S}^N.$$

- Macrostates:

$$P = (p_{ik})_{\substack{1 \le i \le M \\ 1 \le k \le K}} \in [0,1]^{MK}, \sum_{k=1}^{K} p_{ik} = 1,$$

$$\nu_{ik}[\hat{\omega}] = \sum_{\alpha=1}^{n} \delta_{\omega_{i\alpha}, \sigma_k},$$

$$\mathfrak{M}(P) = \{\hat{\omega} \in \mathfrak{S}^N \mid \forall i, k, \nu_{ik}[\hat{\omega}]/n = p_{ik}\}$$

Number of microstates which realize a given macrostate:

$$W(P) = \mathrm{Card}\, \mathfrak{M}(P) = \prod_{i=1}^{M} \frac{n!}{\prod_{k=1}^{K} (np_{ik})!}$$

Macrostates and global constraints

Coarse-grained vorticity field:

$$\overline{\omega}_i = \frac{1}{n}\sum_{\alpha=1}^{n} \omega_{i\alpha} = \sum_{k=1}^{K} \sigma_k p_{ik}.$$

- The energy does not depend on the microstate but only on the macrostate

$$\mathcal{E}[\hat{\omega}] = \frac{1}{2N^2} \sum_{(i,\alpha)\neq(j,\beta)} G_{i\alpha,j\beta}\omega_{i\alpha}\omega_{j\beta},$$

$$= \frac{1}{2M^2}\sum_{i\neq j} G_{ij}\overline{\omega}_i\overline{\omega}_j + o\left(\frac{1}{n}\right).$$

- For $\hat{\omega} \in \mathfrak{M}(P)$,

$$\nu_k^T[\hat{\omega}] = \sum_{i=1}^{N} \nu_{ik}[\hat{\omega}] = n\sum_{i=1}^{N} p_{ik},$$

Global vorticity distribution constraints:

$$\frac{\nu_k^T[P]}{N} = \gamma_k.$$

The mean-field approach: large deviation of the macrostate probability

Probability of a given macrostate P with energy E:

$$\mathrm{Prob}(P) = \frac{\mathrm{Card}\,\mathfrak{M}(P)}{\mathrm{Card}\,\Lambda_N(E, \Delta E)} = \frac{W(P)}{\Omega_N(E, \Delta E)},$$

$$\frac{1}{N}\ln \mathrm{Prob}(P) = \underbrace{-\frac{1}{M}\sum_{i=1}^{M}\sum_{k=1}^{K} p_{ik}\ln p_{ik}}_{\mathscr{S}_{M,K}[P]} - S(E) + o(1).$$

where we have used the Stirling approximation: $\ln n! = n\ln n - n + O(\ln n)$ as $n \to +\infty$.
In other words,

$$\mathrm{Prob}\, P \underset{N\to+\infty}{\sim} e^{N(\mathscr{S}_{M,K}[P]-S(E))}.$$

This is a *large deviation property* (level 2, Sanov's theorem).

The mean-field approach: thermodynamic limit

Microstates

$$\hat{\omega} = (\omega_{i\alpha})_{\substack{1\leq i\leq M\\1\leq\alpha\leq n}} \in \mathfrak{S}^N \underset{n,M,K\to+\infty}{\longrightarrow} \omega(\mathbf{r}) \in L^2(\mathcal{D})$$

Macrostates

$$P = (p_{ik})_{\substack{1\leq i\leq M\\1\leq k\leq K}} \in [0,1]^{MK} \underset{n,M,K\to+\infty}{\longrightarrow} \rho(\sigma,\mathbf{r})$$

$$\forall i \in [\![1,M]\!], \sum_{k=1}^{K} p_{ik} = 1 \underset{n,M,K\to+\infty}{\longrightarrow} \forall \mathbf{r} \in \mathcal{D}, \int_{\mathbb{R}} \rho(\sigma,\mathbf{r})d\sigma = 1$$

$$\overline{\omega}_i = \frac{1}{n}\sum_{\alpha=1}^{n}\omega_{i\alpha} = \sum_{k=1}^{K}\sigma_k p_{ik} \underset{n,M,K\to+\infty}{\longrightarrow} \overline{\omega}(\mathbf{r}) = \int_{\mathbb{R}} \sigma\rho(\sigma,\mathbf{r})d\sigma$$

$$\mathscr{S}_{M,K}[P] = -\frac{1}{M}\sum_{i=1}^{M}\sum_{k=1}^{K} p_{ik}\ln p_{ik} \underset{n,M,K\to+\infty}{\longrightarrow} \mathscr{S}[\rho] \equiv -\int_{\mathcal{D}} d\mathbf{r} \int_{\mathbb{R}} d\sigma\rho(\sigma,\mathbf{r})\ln\rho(\sigma,\mathbf{r})$$

Constraints

$$\frac{1}{2}\sum_{i,j=1}^{M} G_{ij}\overline{\omega}_i\overline{\omega}_j = E \underset{n,M,K\to+\infty}{\longrightarrow} \mathscr{E}[\rho] \equiv \frac{1}{2}\int_{\mathcal{D}^2} d\mathbf{r}d\mathbf{r}' G(\mathbf{r},\mathbf{r}')\overline{\omega}(\mathbf{r})\overline{\omega}(\mathbf{r}') = E$$

$$\forall k \in [\![1,K]\!], \frac{1}{M}\sum_{i=1}^{M} p_{ik} = \gamma(\sigma_k) \underset{n,M,K\to+\infty}{\longrightarrow} \forall \sigma \in \mathbb{R}, \mathscr{D}_\sigma[\rho] \equiv \int_{\mathcal{D}} \rho(\sigma,\mathbf{r})d\mathbf{r} = \gamma(\sigma)$$

The mean-field approach: variational problem

Equilibrium states = most probable macrostates. They must minimize the large deviation rate function, while satisfying the global constraints.

Microcanonical variational problem

$$S(E,\gamma) = \max_{\rho}\{\mathscr{S}[\rho] \mid \mathscr{E}[\rho] = E, \forall \sigma \in \mathbb{R}, \mathscr{D}_\sigma[\rho] = \gamma(\sigma)\}.$$

Critical points:

$$0 = \delta\mathscr{S} - \int_{\mathcal{D}} d\mathbf{r}\zeta(\mathbf{r}) \int_{\mathbb{R}} d\sigma\delta\rho(\sigma,\mathbf{r}) - \beta\delta\mathscr{E} - \int_{\mathbb{R}} d\sigma\alpha(\sigma) \int_{\mathcal{D}} d\mathbf{r}\delta\rho(\sigma,\mathbf{r}),$$

$$\rho(\sigma,\mathbf{r}) = \exp(-1 - \zeta(\mathbf{r}) - \alpha(\sigma) - \beta\sigma\overline{\psi}(\mathbf{r})),$$

$$\rho(\sigma,\mathbf{r}) = \frac{e^{-\beta\sigma\overline{\psi}(\mathbf{r})-\alpha(\sigma)}}{\mathcal{Z}_{\beta,\alpha}(\overline{\psi}(\mathbf{r}))} \qquad \text{(Gibbs states)},$$

with

$$\overline{\omega} = -\Delta\overline{\psi}, \qquad \mathcal{Z}_{\beta,\alpha}(u) = \int_{\mathbb{R}} e^{-\beta\sigma u-\alpha(\sigma)} d\sigma.$$

The mean-field equation for the coarse-grained vorticity field

Mean-field equation:

$$\overline{\omega}(\mathbf{r}) = \frac{1}{\mathcal{Z}_{\beta,\alpha}(\overline{\psi}(\mathbf{r}))} \int_{\mathbb{R}} d\sigma \sigma e^{-\beta\sigma\overline{\psi}(\mathbf{r})-\alpha(\sigma)},$$

$$\overline{\omega}(\mathbf{r}) = F_{\beta,\alpha}(\overline{\psi}(\mathbf{r})), \qquad \text{with } F_{\beta,\alpha}(u) = -\frac{1}{\beta}\frac{d \ln \mathcal{Z}_{\beta,\alpha}(u)}{du}.$$

In particular, the equilibrium coarse-grained vorticity field is a stationary solution of the 2D Euler equation. Further, it is dynamically stable.
In general, this equation is difficult to solve:

- It is a nonlinear partial differential equation.
- Analytic computation of the partition function $\mathcal{Z}_{\beta,\alpha}(u)$ is rarely possible.
- We need to relate *a posteriori* the Lagrange parameters $\beta, \alpha(\sigma)$ to the conserved quantities $E, \gamma(\sigma)$.

Numerical methods: relaxation equations[18], Turkington-Whitaker algorithm[19],...

[18] R. Robert and J. Sommeria (1992). *Phys. Rev. Lett.* P.-H. Chavanis (2009). *Eur. Phys. J. B*
[19] B. Turkington and N. Whitaker (1996). *SIAM J. Sci. Comput.*

The linear mean-field equation

When the function $F_{\beta,\alpha}$ is linear, the mean-field equation can be solved analytically. When does this happen?

- "Strong mixing" limit[20]: $\beta \to 0$, or "low-energy" limit: $\overline{\psi} \to 0$.
- Energy-enstrophy variational problem
- Subclass of the full MRS equilibrium states[21].

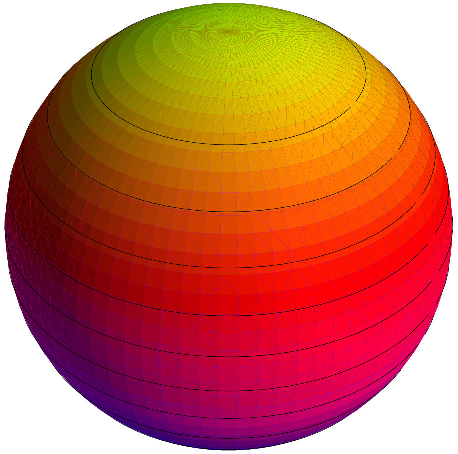
Then analytical computations are possible, by introducing the eigenmodes of the Laplacian on the domain $\mathcal{D}$.

[20] P.-H. Chavanis and J. Sommeria (1996). *J. Fluid Mech.*
[21] F. Bouchet (2008). *Physica D*

Equilibrium flows on the sphere

Stable equilibrium states[22]

- Solid body rotations: $\psi = \Omega_* \cos\theta$



Solid body rotation

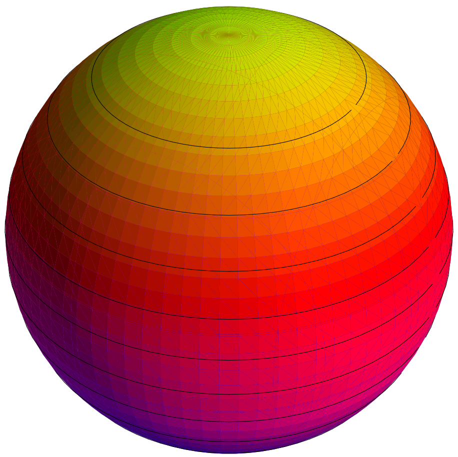
For a solid-body rotation, the invariants E and L are not independent, they must satisfy $E = 3L^2/4 \equiv E^*(L)$.

[22] C. Herbert et al. (2012b). *J. Stat. Mech.*

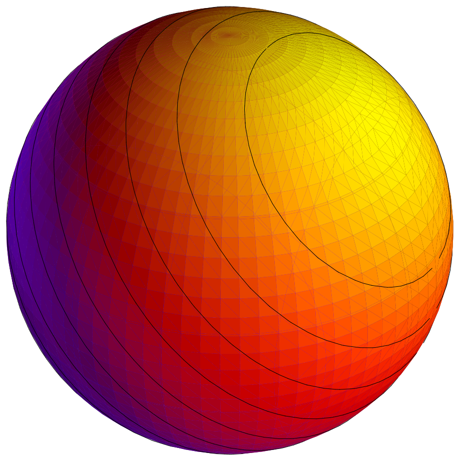
Equilibrium flows on the sphere

Stable equilibrium states[22]

- Solid body rotations: $\psi = \Omega_* \cos\theta$
- Dipoles: $\psi = \Omega_* \cos\theta + \sqrt{3(E - E^*(L))} \sin\theta \cos(\phi - \phi_0)$



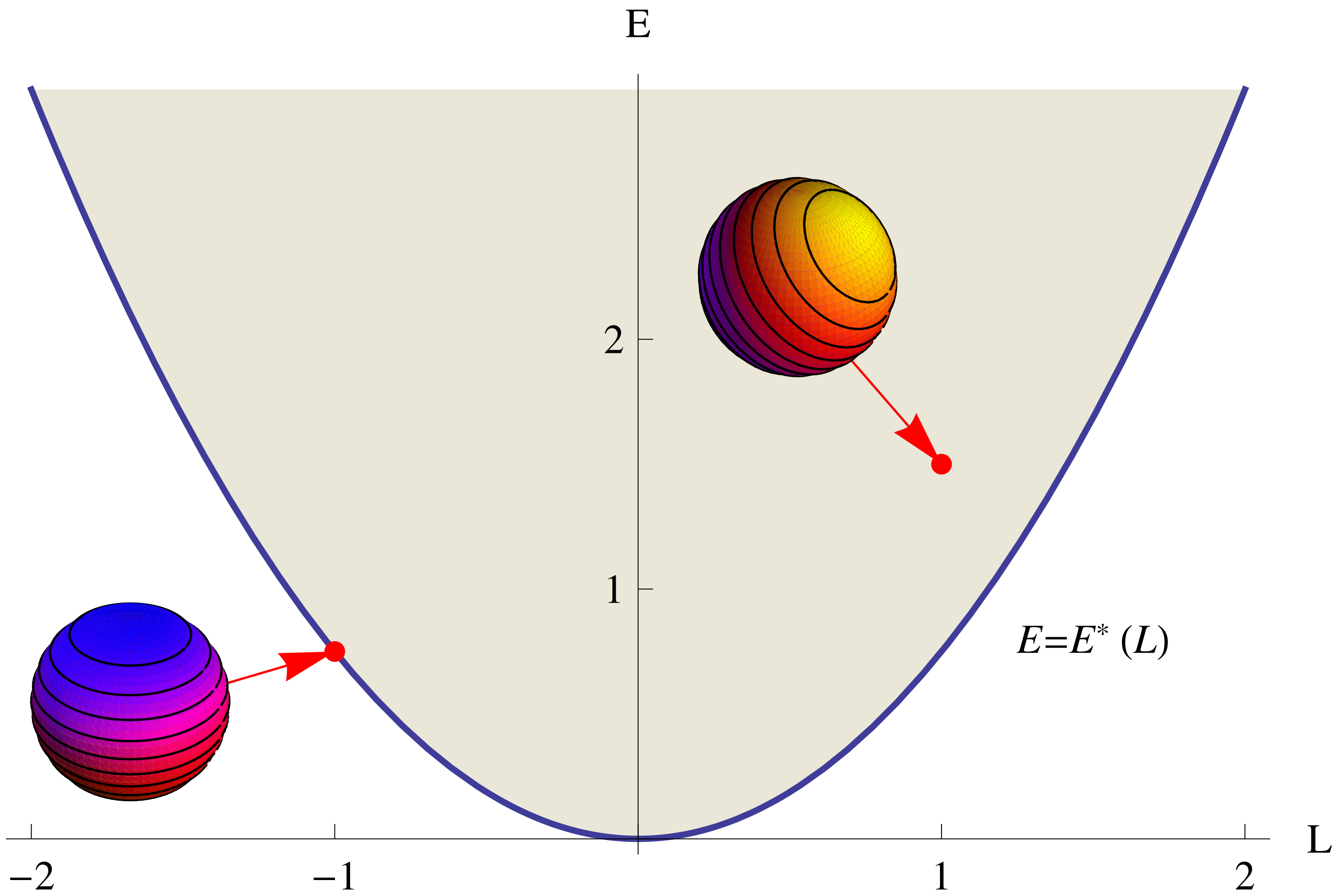
Solid body rotation



Dipole

[22] C. Herbert et al. (2012b). *J. Stat. Mech.*

Microcanonical Phase Diagram



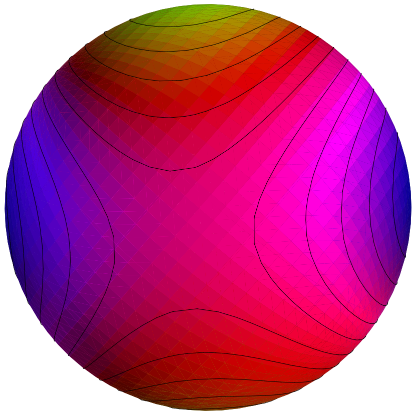
Second-order phase transition with spontaneous symmetry breaking.[23]

[23] C. Herbert et al. (2012b). *J. Stat. Mech.*

Geometrical Refinement

Stable equilibrium states

- Solid body rotations
- Dipoles
- Quadrupoles :
  $\psi_\infty = \psi_{20}(3\cos^2\theta - 1) + \psi_{21}\sin(2\theta)\sin(\phi - \phi_1) + \psi_{22}\sin^2\theta\sin(2(\phi - \phi_2))$



Theoretical Equilibrium: Quadrupole[24]

[24] C. Herbert (2013). *J. Stat. Phys.*

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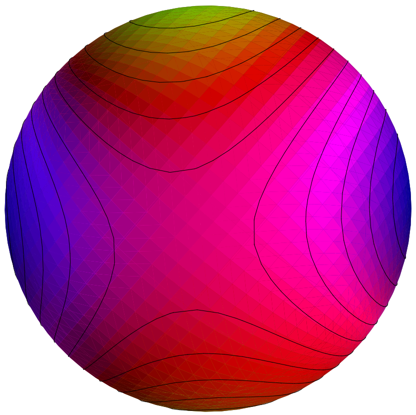

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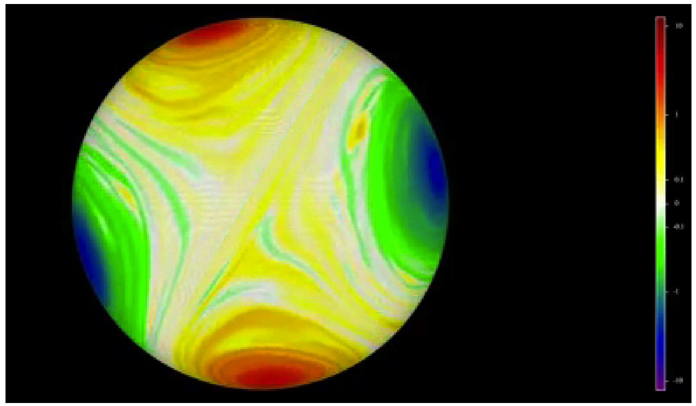

DNS Final State[25]

[24] C. Herbert (2013). *J. Stat. Phys.*

[25] B. Marston (2011). *Physics*; W. Qi and J. B. Marston (to appear). *J. Stat. Mech.*

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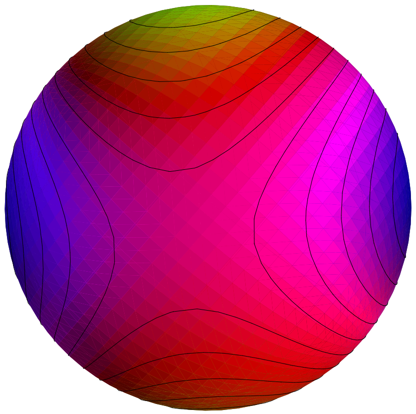

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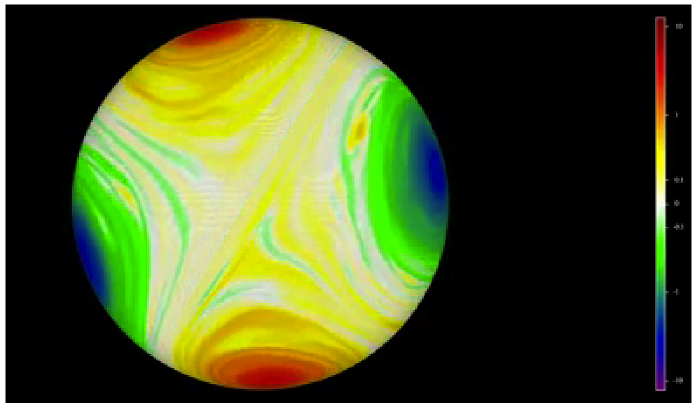


DNS Final State[25]

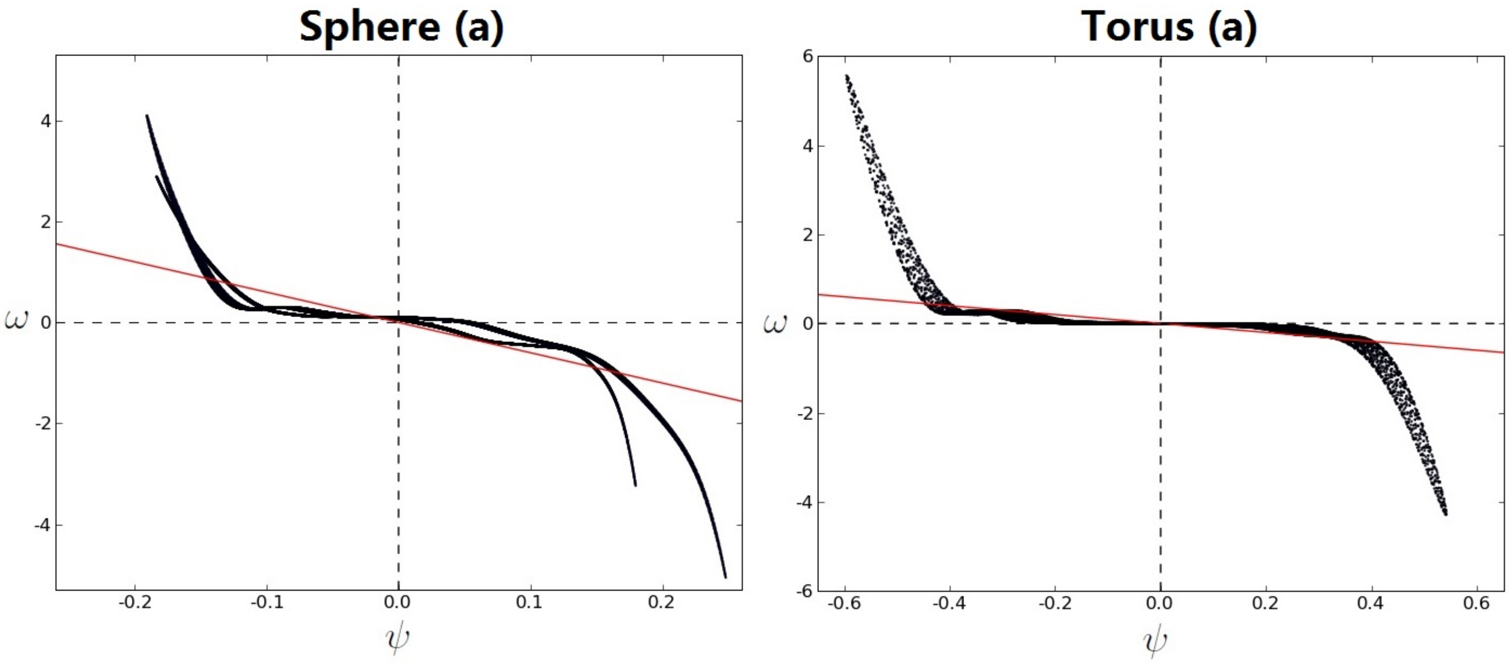
Time dependent macrostates.

[24] C. Herbert (2013). *J. Stat. Phys.*

[25] B. Marston (2011). *Physics*; W. Qi and J. B. Marston (to appear). *J. Stat. Mech.*

Role of the higher-order invariants

Comparing the equilibrium states with numerical simulations[26]:



[26] W. Qi and J. B. Marston (to appear). *J. Stat. Mech.*

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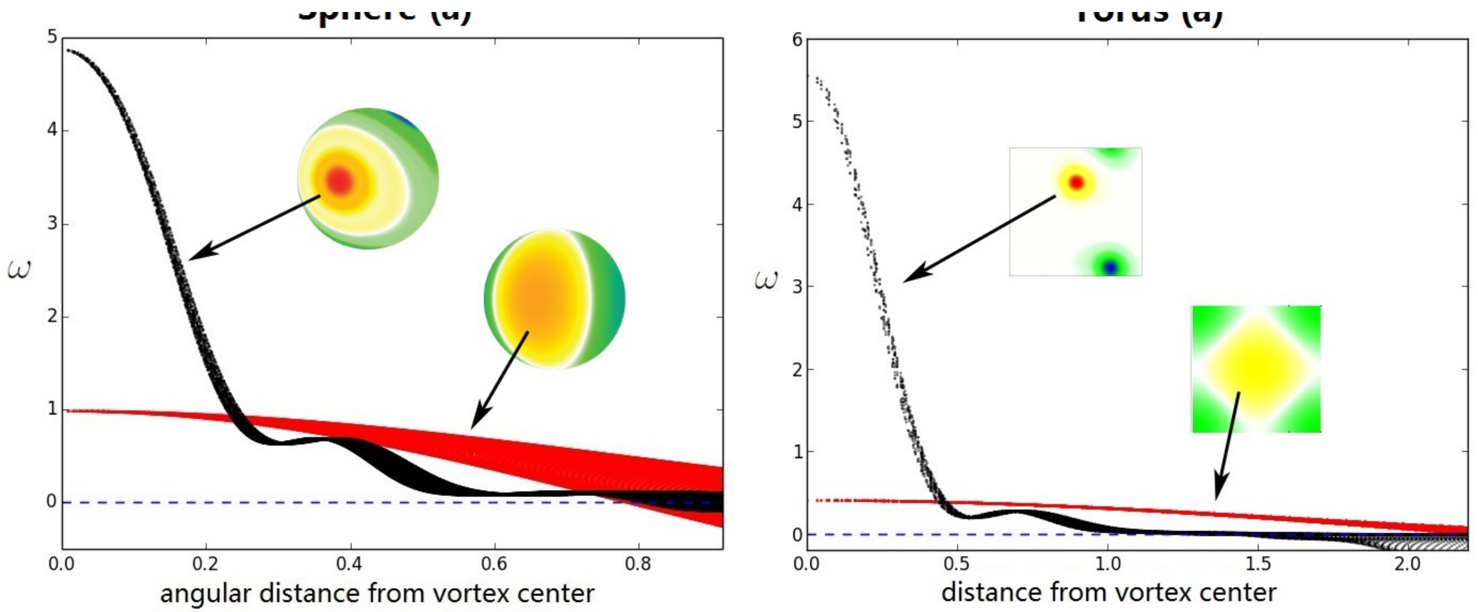

Non-linear $\omega - \psi$ relationships are associated to strong vorticity gradients: e.g. the two-level system[27].

[26] W. Qi and J. B. Marston (to appear). *J. Stat. Mech.*

[27] F. Bouchet and J. Sommeria (2002). *J. Fluid Mech.* A. Venaille and F. Bouchet (2011a). *J. Phys. Oceanogr.*

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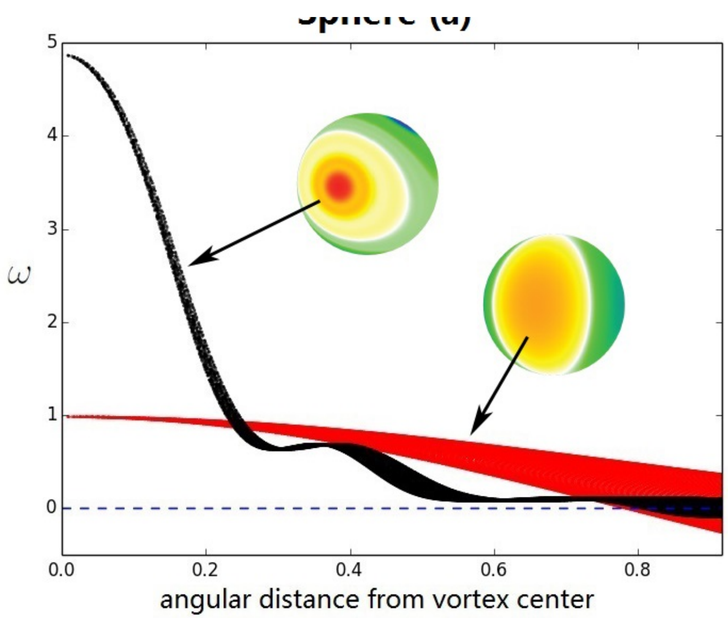

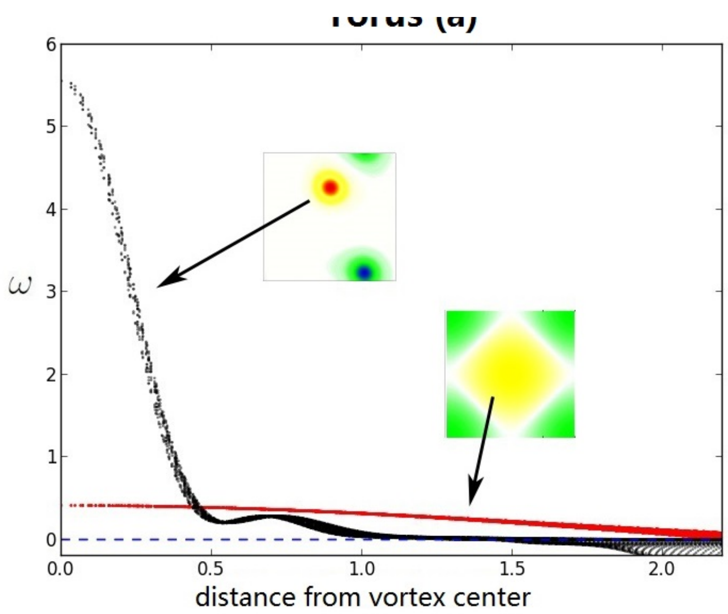

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Perturbative expansion leads to core sharpening.

[26] W. Qi and J. B. Marston (to appear). *J. Stat. Mech.*

[27] F. Bouchet and J. Sommeria (2002). *J. Fluid Mech.* A. Venaille and F. Bouchet (2011a). *J. Phys. Oceanogr.*

Summary

How is energy transferred across scales in geophysical turbulence?

Energy cascade for Rotating-Stratified flows

- Statistical Mechanics in the restricted ensemble provides support to the idea that inverse cascades in rotating and rotating-stratified may exist due to the slow modes, even though the slow manifold is not rigorously invariant.
- On the contrary, it is predicted that the slow modes of stratified turbulence cascade energy downscale, because of the presence of the shear modes which do not contribute to potential enstrophy.

Summary

How are large-scale coherent structures formed in geophysical flows?

RSM mean-field theory for quasi-2D flows

- The mean-field theory allows one to compute statistical equilibrium states, which correspond to observed large-scale structures.
- Energy and enstrophy conservation yield complete condensation of the energy in the gravest modes[28]. Higher-order Casimir invariants[29] or geometrical constraints[30] can prevent the condensation from being complete.
- Rotation can also arrest the cascade and lead to the formation of zonal flows, through the effect of waves[31].
- Theoretical aspects: large deviations, non-equivalence of the statistical ensembles[32].

[28] F. Bouchet and M. Corvellec (2010). *J. Stat. Mech.*

[29] R. Abramov and A. J. Majda (2003). *Proc. Natl. Acad. Sci. U.S.A.*

[30] C. Herbert (2013). *J. Stat. Phys.*

[31] P. B. Rhines (1975). *J. Fluid Mech.*

[32] R. S. Ellis et al. (2000). *J. Stat. Phys.* F. Bouchet (2008). *Physica D*; P.-H. Chavanis (2009). *Eur. Phys. J. B*; A. Venaille and F. Bouchet (2011b). *J. Stat. Phys.* C. Herbert et al. (2012a). *Phys. Rev. E*

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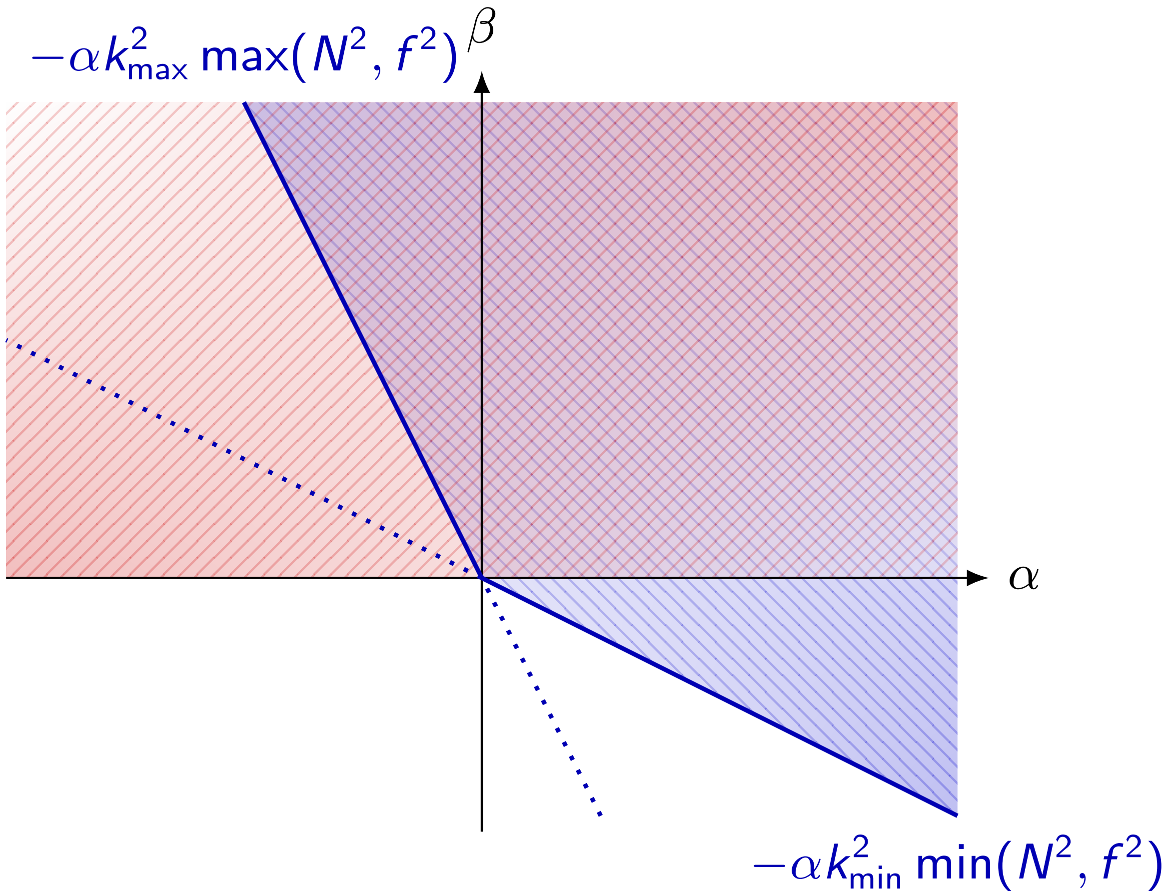

Accessible thermodynamic space for rotating-stratified flows, waves (red) and slow manifold (blue).

The helical decomposition for the 3D Euler equation

Euler equations for 3D homogeneous isotropic turbulence:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P,$$
$$\nabla \cdot \mathbf{u} = 0.$$

Helical decomposition in Fourier space[33]: $\nabla \times \mathbf{h}_\pm(\mathbf{k}) = \pm k \mathbf{h}_\pm(\mathbf{k})$,

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} [u_+(\mathbf{k})\mathbf{h}_+(\mathbf{k}) + u_-(\mathbf{k})\mathbf{h}_-(\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}},$$
$$\boldsymbol{\omega}(\mathbf{x}) = \nabla \times \mathbf{u} = \sum_{\mathbf{k}} k[u_+(\mathbf{k})\mathbf{h}_+(\mathbf{k}) - u_-(\mathbf{k})\mathbf{h}_-(\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}}$$

Automatically enforces incompressibility: $\mathbf{k} \cdot \mathbf{h}_\pm(\mathbf{k}) = 0$.
Energy and Helicity:

$$E = \frac{1}{2} \int \mathbf{u}(\mathbf{x})^2 d\mathbf{x} = \frac{1}{2} \sum_{\mathbf{k}} [|u_+(\mathbf{k})|^2 + |u_-(\mathbf{k})|^2],$$
$$H = \frac{1}{2} \int \mathbf{u}(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x}) d\mathbf{x} = \frac{1}{2} \sum_{\mathbf{k}} k[|u_+(\mathbf{k})|^2 - |u_-(\mathbf{k})|^2].$$

[33] A Craya (1958). *Publ. Sci. Tech. Ministère de l'Air*; J. R. Herring (1974). *Phys. Fluids*; F. Waleffe (1992). *Phys. Fluids A*